Calculus II: For Physical Sciences and Engineering (AS.110.109)

Lecture: 3

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Textbook Sections: 5.5

⚠ Mention HW expectations ⊕ test run on Gradescope

Topic for today: u-substitution

As we explored in the last lecture, its very easy to evaluate an integral (definite or indefinite) provided we can find a anti-derivative for the integrand.

For functions like x^n, $\cos x$, $\sin x$, e^x, etc, this isnt so hard to do on sight. However, its very easy to run into integrands which don't have such a friendly form:

$$x^3\cos(x^4+2),\ \sqrt{1-x^2},\ e^{x^2},\ \frac{1}{1+x^4},\ x\cdot\sin x,\ \text{etc}$$

In situations like these, finding an anti-derivative can be a lot trickier — in some cases, it'll turn out to even be impossible to find an anti-derivative that you can write down in terms of "normal functions."

SO... we need to develop some new techniques for manipulating integrals, so that we can put them into forms we can make sense of.

A few useful techniques that we will develop are built upon the idea of "doing differentiation rules in reverse." As a first example of this idea, we introduce the technique of "u-substitution"

Guiding example: Think about the integral $\int 2x\sqrt{1+x^2}\,dx$.

Key thing to notice: A function *and* its derivative show up in the integrand:

$$\int 2x\sqrt{1+x^2}\,dx.$$

Let's let $u = 1+x^2$, so that $u' = \frac{du}{dx} = 2x$.

If we think about the language of differentials, we have that

$$du = 2x\,dx,$$

which appears in the integral! So, if we think of

$$u = 1+x^2$$

as a new variable, and replace the differential $2x\,dx$ with du, we can write the integral in a new form:

$$\int 2x\sqrt{1+x^2}\,dx = \int \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}} + C$$

Then filling in for $u = 1+x^2$ gives the final answer (i.e., in the original variables):

$$\int 2x\sqrt{1+x^2}\,dx = \frac{2}{3}(1+x^2)^{\frac{3}{2}} + C.$$

So what is going on?

Recall the **chain rule**: If $h(x) = f(g(x))$ is a composition of differentiable functions, then

$$h'(x) = f'(g(x)) \cdot g'(x).$$

Key idea: Let's "undo" this by taking an indefinite integral:

$$\int f'(g(x))g'(x)\,dx = \int h'(x)\,dx = h(x) + C$$
$$= f(g(x)) + C.$$

Thus, if our integrand happens to be in the form

$$\varphi(g(x))\,g'(x)$$

[let's replace $f'(x)$ from earlier with a more general function $\varphi(x)$ here]

then "doing the chain rule in reverse" tells us how to solve the integral, once we find an antiderivative $\Phi(x)$ for $\varphi(x)$.

Symbolically, we can summarize this with the change of variables as above

$$\int \varphi(g(x))\,g'(x)\,dx = \int \varphi(u)\,du = \Phi(u)+C = \Phi(g(x))+C$$

Here's the corresponding statement for definite integrals:

If g is differentiable on $[a,b]$, g' is cts, and φ is cts on the range of g, then

$$\int_a^b \varphi(g(x))\,g'(x)\,dx = \int_{g(a)}^{g(b)} \varphi(u)\,du$$

⚠ When we change variables from x to $u=g(x)$, we also need to change the limits of integration:

$$\{a\le x\le b\} \xrightarrow[u=g(x)]{} \text{u lives between } g(a) \text{ and } g(b)$$

⚠ Note that $g(a)$ could be bigger than $g(b)$! For example:

$$\int_0^1 2xe^{1-x^2}\,dx \underset{\substack{u=1-x^2\\ du=-2x\,dx}}{=} -\int_1^0 e^u\,du = \int_0^1 e^u\,du = e-1$$

Lastly for today,

Exercises:

1) $\int e^{2x}\,dx$

2) $\int_0^{\sqrt{\pi}} x\cos(x^2)\,dx$

3) $\int \tan(x)\,dx$

4) $\int \left(\frac{x}{\sqrt{1-4x^2}} + x\right) dx$

5) $\int \sqrt{1+x^2}\,x^5\,dx = \int \sqrt{1+x^2}\,x^4 \cdot x\,dx \qquad u = 1+x^2$